An innovative idea presented to honor an innovative demographer, Anatoly Vishnevsky

FIRST DEMOGRAPHIC READINGS IN MEMORY OF ANATOLY VISHNEVSKY
Demographic horizons of Russia and the world in the medium and long term perspective
National Research University Higher School of Economics, 9-11 November 2021

James W. Vaupel

Interdisciplinary Centre on Population Dynamics, University of Southern Denmark

9 November 2021

Mortality as a function of survival

- Talk today based on a draft by Jesus-Adrian Alvarez and James W. Vaupel that we will soon submit for publication and a draft I wrote that was just submitted for publication.
- A longer article by Trifon Missov, Jesus-Adrian Alvarez, Annette Baudisch and James W. Vaupel is being prepared.

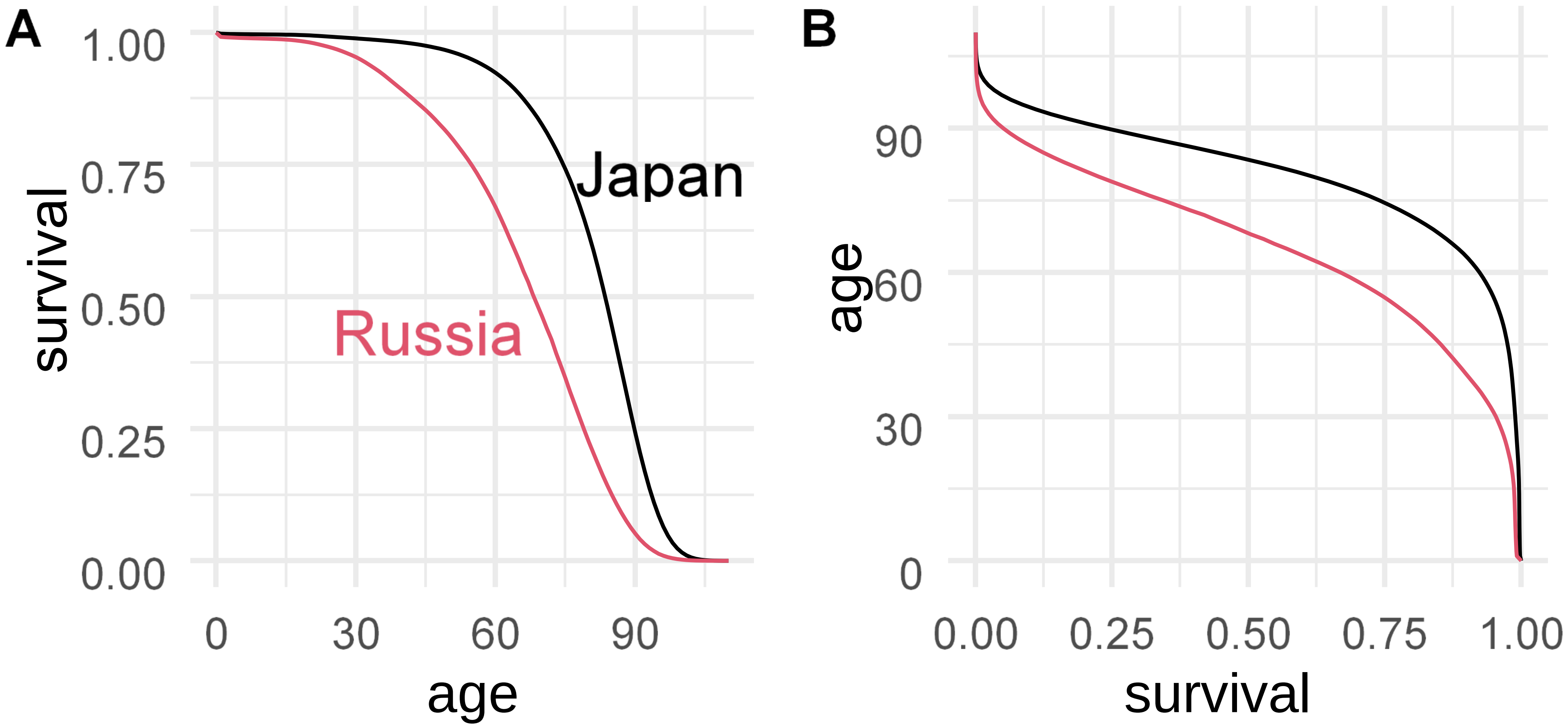
Mortality as a function of survival

Everyone has a chronological age.

Everyone in a population has a survival age,
the proportion of the person's birth cohort
that is still alive.

Mortality can be studied over age.
Mortality can be studied over survival age.

Males 2014



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survival
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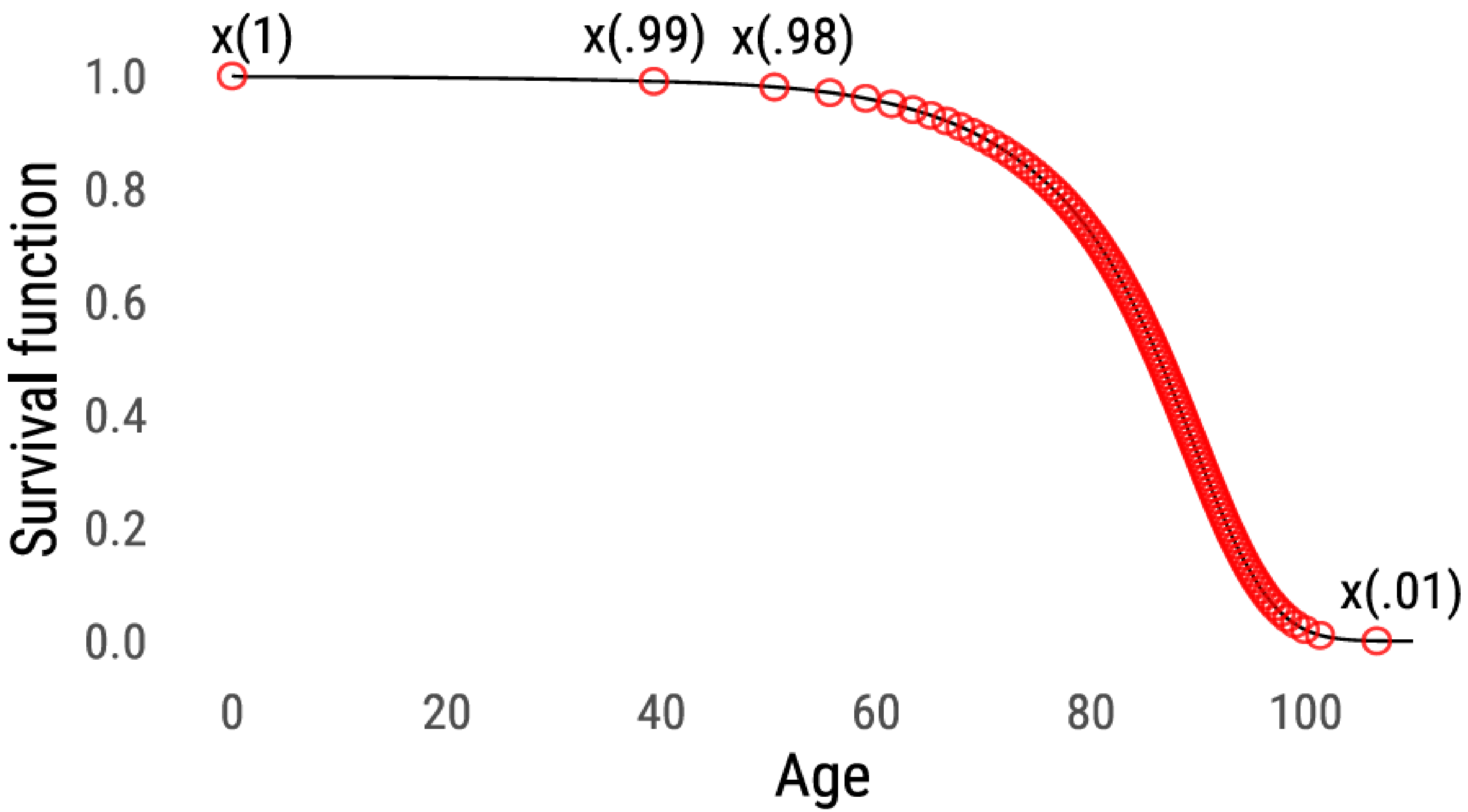
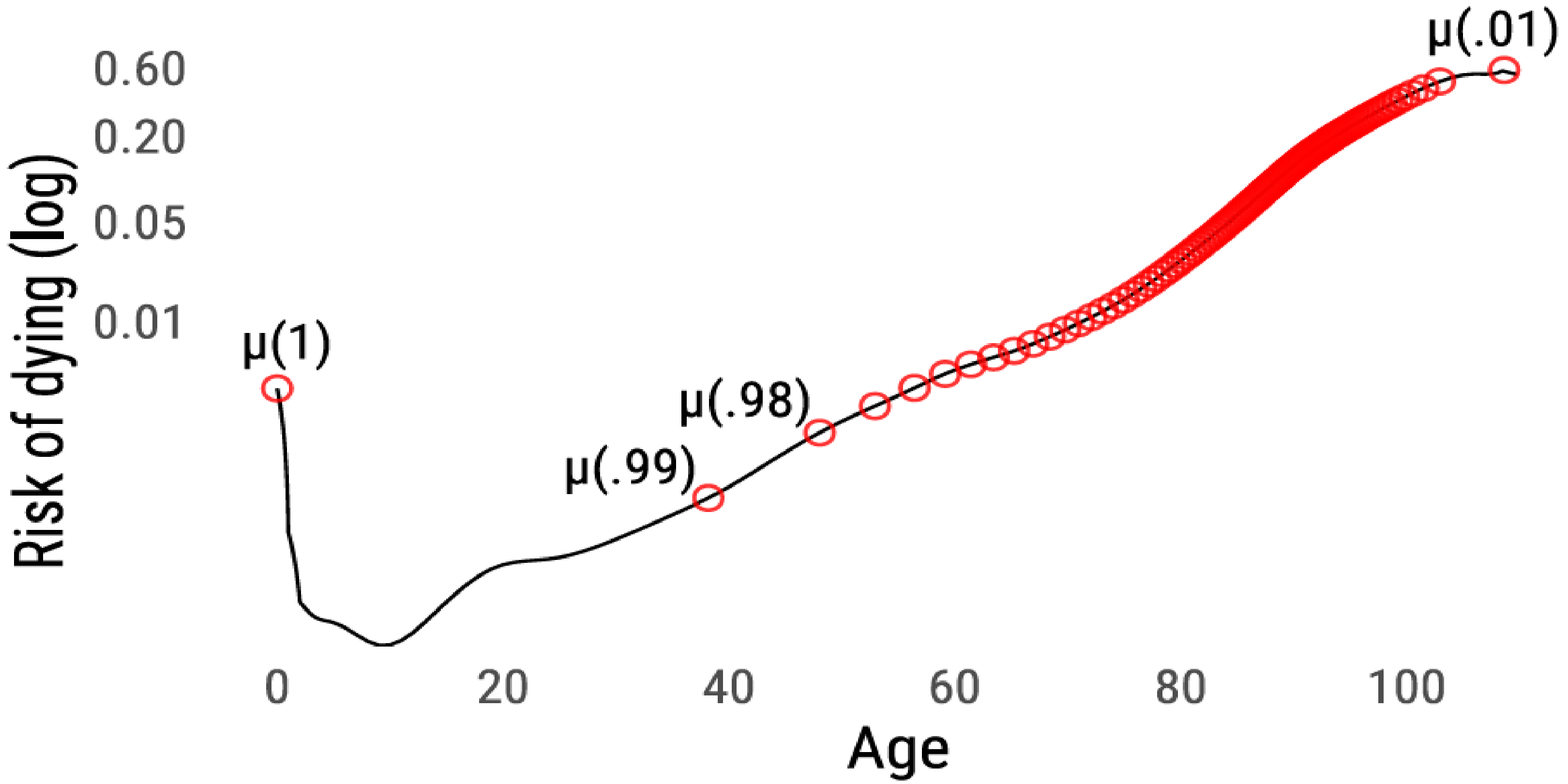
survival

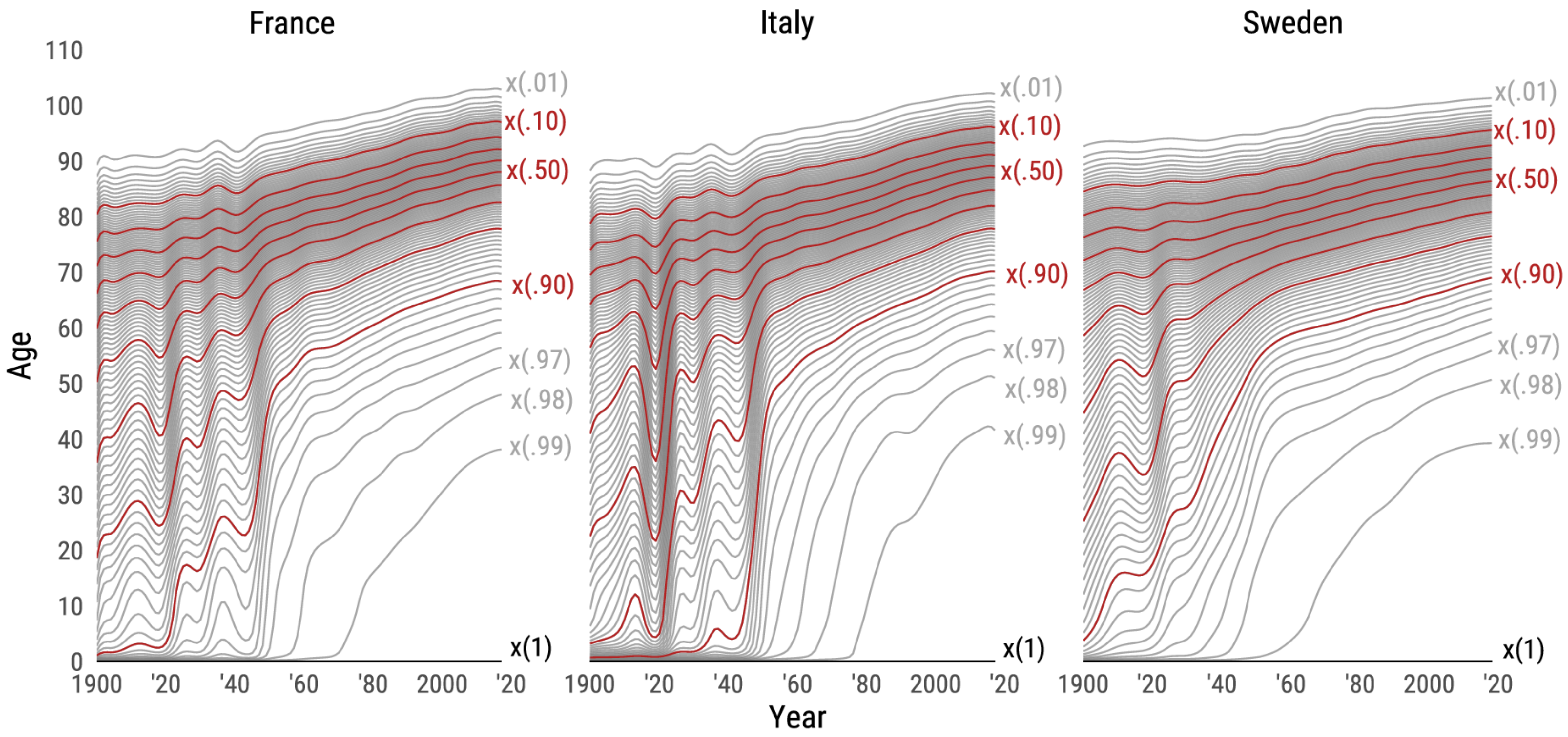
Survival curve (Sw. F 2018) with x(s) marked



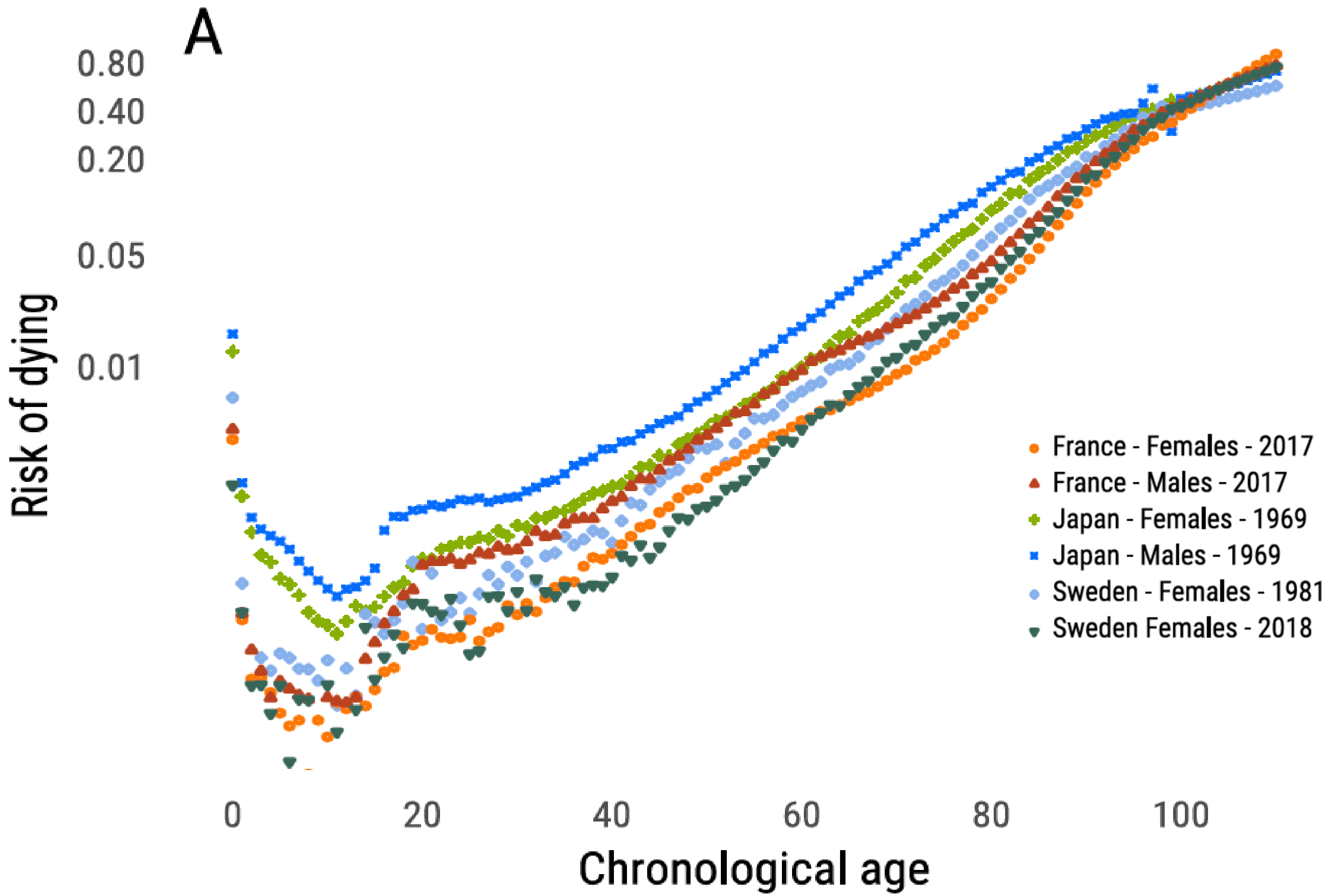
Force of mortality (Sw. F 2018) with s-ages marked



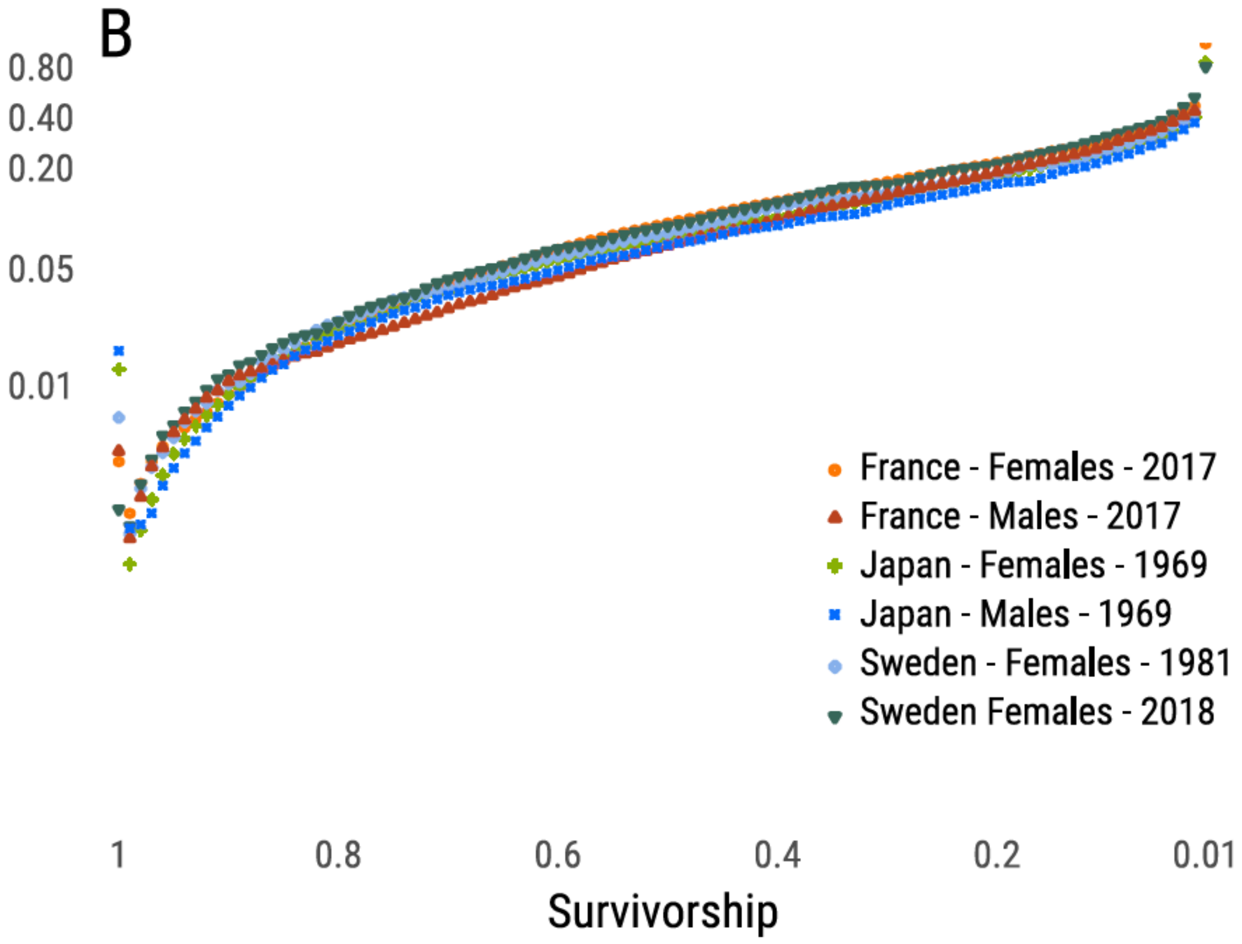
Survivorship ages for France, Italy and Sweden. Females, 1900-2018. Red lines indicate deciles



Log age-specific force of mortality by age



Log force of mortality by survival age



Survival age for Gompertz mortality

If $$\mu(x) = ae^{bx} = be^{b(x-M)}$$

then $$s(x) = e^{-e^{-bM}(e^{bx}-1)}.$$

Hence $$x(s) = \ln(1 - e^{bM}\ln(s))/b.$$

If s is 50%, b is 0.14 and M is 90, then the median age at death is 87.4.

Gompertz median as a function of the mode

$$x(s) = \ln(1 - e^{bM} \ln(s))/b$$

$$X_{.5} = \ln(1 - e^{bM} \ln(0.5))/b$$

For values of b around 0.1 and values of M exceeding 70, $(e^{bM})(-\ln(0.5)) \gg 1$, so

$$X_{.5} \approx \frac{\ln[(e^{bM})(-\ln(0.5))]}{b} = M + \ln(-\ln(0.5))/b \approx M - 0.3665/b.$$

The density of deaths as a function of survival

Consider the density of deaths: $d(x) = -\frac{ds(x)}{dx}$.

Note *d(x)* is NOT the same as *dx*.

When mortality is viewed as a function of survivorship:

$$d\big(x(s)\big) = -\frac{ds}{dx(s)}.$$

Since $d\big(x(s)\big) \equiv d(s)$, this leads to $d(s) = \frac{-1}{dx(s)/ds}$.

In the Gompertz case, $d(s) = b(e^{-bM} - \ln s))s$.

The force of mortality as a function of survival

The force of mortality as a function of age is $\mu(x) = d(x)/s(x)$.

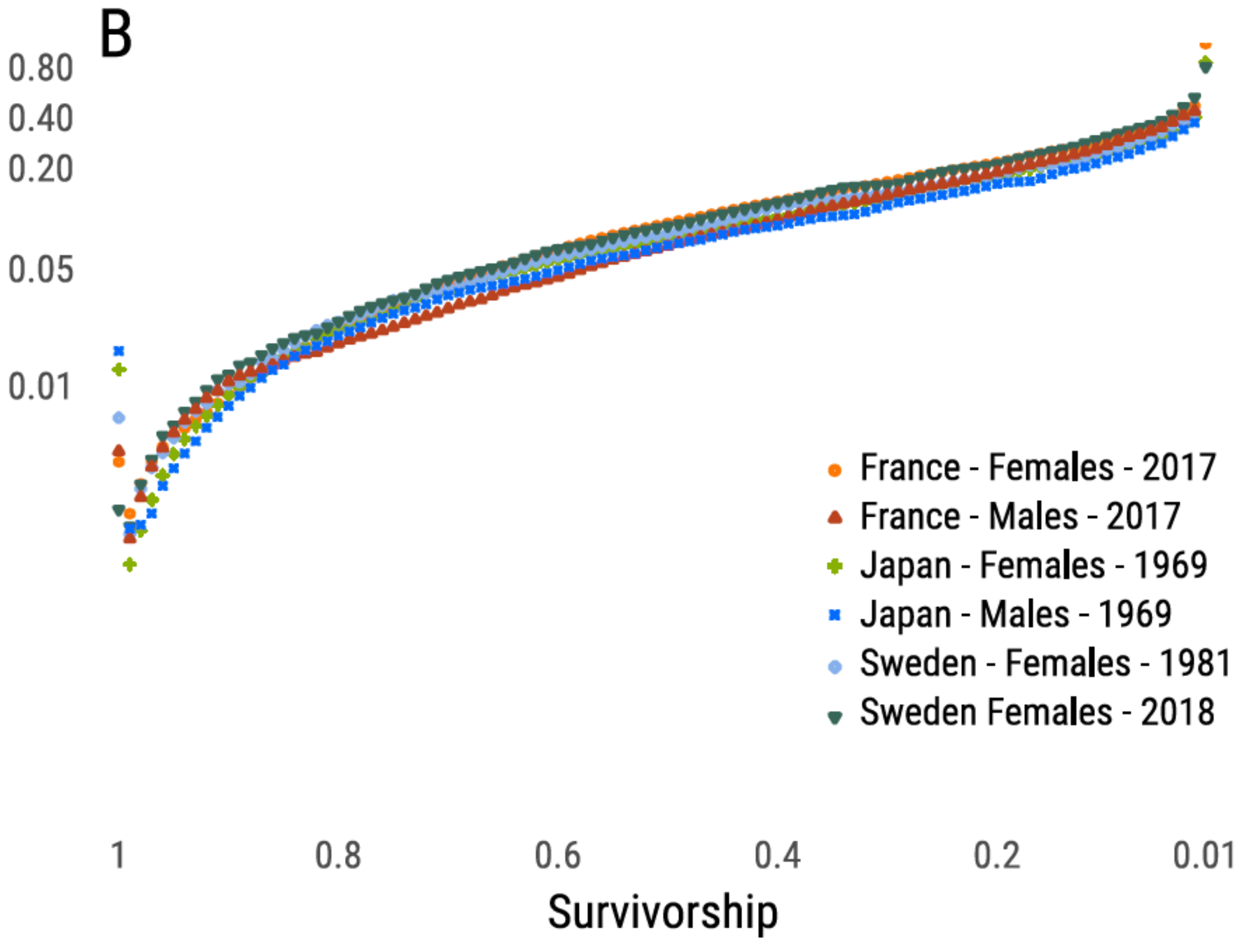
As a function of survivorship, it is $\mu(s) = \frac{d(s)}{s} = \frac{1}{(\frac{dx(s)}{ds})s}$.

For Gompertz mortality, $\mu(s) = b(e^{-bM} - \ln s)$.

In many applications, e^{-bM} will be very small. For instance, $e^{-(.14)(90)} =$ 0.000003. If so,

$$\mu(s) = -b \ln s$$

Log force of mortality by survival age



Mortality as a Function of Survival—formulas if Gompertz

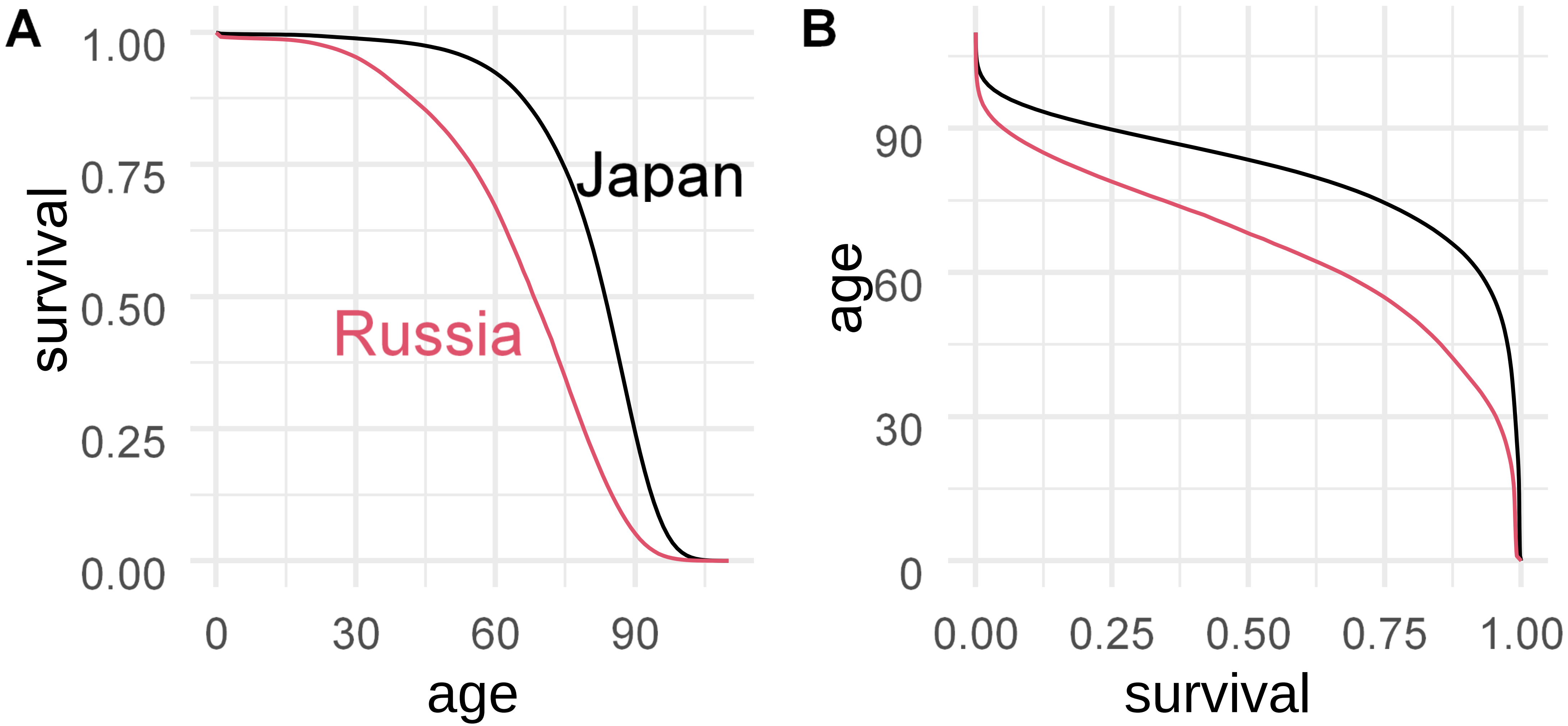
Quantity	Units	Symbol	x	s
Age	years	x	x	$\ln\left(1-\ln(s)\,e^{bM}\right)$
Survival	none	s	$e^{-e^{-bM}(e^{bx}-1)}$	s
Hazard	/year	μ	$be^{b(x-M)}$	$b(e^{-bM}-\ln(s))$
Cum. Hazard	none	H	$e^{-bM}(e^{bx}-1)$	$-\ln(s)$
Death density	/year	d	$be^{b(x-M)}e^{-e^{-bM}(e^{bx}-1)}$	$sb(e^{-bM}-\ln(s))$
Life expectancy	years	e_o	$\int_0^{\infty} e^{-e^{-bM}(e^{bx}-1)}\,dx$	$\int_0^{1}\ln\left(1\ln(s)\,e^{-bM}\right)ds/b$

Life Expectancy

Survival is usually plotted against age x and life expectancy is usually viewed as the area under the $s(x)$ curve.

For any strictly declining survival curve, including the Gompertz survival curve, the area between the survival curve and the vertical axis equals the area between the survival curve and the horizontal axis.

Males 2014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1.00
survival

Gompertz life expectancy in terms of the mode

$$e_o = \int_0^1 \ln(1 - e^{bM}\ln(s))ds/b\ .$$

If e^{bM} is large, e.g., if b is 0.14 and M is 90, then at values of s when $-e^{bM}\ln(s)$ is large, the constant 1 is unimportant. Therefore,

$$e_o \approx \int_0^1 \ln\left((e^{bM})(-\ln(s))\right)ds/b = M - \int_0^1 \ln(-\ln s)ds/b.$$

The integral on the right is the Euler-Mascheroni constant, 0.5772…. Hence,

$$e_o \approx M - 0.5772/b.$$

Gompertz mode, median and mean

Gompertz curve: $\mu(x) = ae^{bx} = be^{b(x-M)}$.

Because $\mu(M) = b$,

$$M = \ln(b/a)/b.$$

And, as shown,

$$X_{.5} \approx M - 0.3665/b$$
$$e_o \approx M - 0.5772/b.$$

Anatoly Vishnevsky

- A great Demographer
- Who made major contributions to demography in Russia and globally
- Who loved
  - demographic facts and
  - demographic ideas